Kuwait University	**Math 101**	**Date:**	**January 20, 2009**
Dept. of Math. & Comp. Sci.	**Final Exam**	**Duration:**	**Two Hours**

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions. Each question weighs 4 points.

1. Evaluate the following limits, if they exist:

(a) $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$ (b) $\lim_{\theta \to 0} \frac{2\sin\theta - \sin 2\theta}{\theta^2}$

2. Let

$$f(x) = \begin{cases} \dfrac{x^2-k}{x^2+1} & , \text{ if } x \geq 0, \\ \dfrac{x^3-k+1}{x^2+2} & , \text{ if } x < 0. \end{cases}$$

(a) Find the value of k such that f is continuous at $x = 0$.

(b) Is f continuous at $x = 3$? Justify your answer.

3. Find $f'(1)$, where $f(x) = \sqrt{1+x^2} - \frac{x^3}{x^4+1}$.

4. Find all numbers c that satisfy the conclusion of the Mean Value Theorem, of the function f on the interval $[-1, 2]$, where $f(x) = x^3 - 2x$.

5. Show that:

$$\int \sin^3 x \sqrt{\cos x}\, dx = \tfrac{2}{7}(\cos x)^{7/2} - \tfrac{2}{3}(\cos x)^{3/2} + C.$$

6. Evaluate:

$$\int_0^3 \sqrt{3-x}\, dx + \int_{-2}^{2} x^3 \cos x\, dx.$$

7. Find the x-coordinates of the points of inflection of the continuous function:

$$f(x) = \int_{-5}^{x^2} \frac{1}{3+t^2}\, dt + \int_1^7 \sqrt{3+t^2}\, dt.$$

8. Show that:

$$12 \leq \int_0^3 \sqrt{x^2+16}\, dx \leq 15.$$

9. Find the area of the region bounded by the curves $y = x - 2$ and $y^2 = x - 2$.

10. The region bounded by the curves $y = \sqrt{x}$ and $y = x^3$ is revolved about:

(a) the line $y = 2$,

(b) the line $x = 2$.

Set up an integral that can be used to find the volume of the resulting solid in each case.

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1. (a) $\lim\limits_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)=\lim\limits_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{(x-2)(x+2)}\right)=\lim\limits_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{1}{x+2}=\boxed{\frac{1}{4}}$.

   (b) $\lim\limits_{\theta\to 0}\frac{2\sin\theta-\sin 2\theta}{\theta^2}=\lim\limits_{\theta\to 0}\frac{2\sin\theta-2\sin\theta\cos\theta}{\theta^2}=\lim\limits_{\theta\to 0}\frac{2\sin\theta(1-\cos\theta)}{\theta^2}=2\left(\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\theta}\right)=$ $2\times 1\times 0=\boxed{0}$.

2. (a) $\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}\frac{x^2-k}{x^2+1}=\boxed{-k}=f(0)$ & $\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}\frac{x^3-k+1}{x^2+2}=\boxed{\frac{-k+1}{2}}$.
   f is continuous at 0 $\Longrightarrow\ \frac{-k+1}{2}=-k\ \Longrightarrow \boxed{k=-1}$.

   (b) <u>At $x=3$</u>: $\lim\limits_{x\to 3}f(x)=\lim\limits_{x\to 3}\dfrac{x^2-k}{x^2+1}=\lim\limits_{x\to 3}\dfrac{x^2-(-1)}{x^2+1}=\boxed{1}=f(3)\Longrightarrow$
   Yes, f is *continuous* at $x=3$. [or, for $x=3>0$, f is a rational function, with nonzero denominator, then f is continuous at $x=3$].

3. $\boxed{f'(x)=\frac{x}{\sqrt{1+x^2}}-\frac{-x^6+3x^2}{(x^4+1)^2}}\Longrightarrow\boxed{f'(1)=\frac{1}{\sqrt{2}}-\frac{1}{2}}$.

4. f is continuous on $[-1,2]$, and differentiable on $(-1,2)$ (f is polynomial). $f'(x)=3x^2-2$, $f(-1)=1$, $f(2)=4$. From the Mean Value Theorem, $\exists\, c\in(-1,2)$ such that $f(2)-f(-1)=f'(c)[2-(-1)]$, *i.e.*, $3c^2-2=1\ \Longrightarrow\ \boxed{c^2=1}\ \Longrightarrow\ c=\pm 1\ \Longrightarrow$ Only, $\boxed{c=1}$ satisfy the conclusion of the Mean Value Theorem of f on $[-1,2]$.$[-1\notin(-1,2)]$.

5. <u>Solution (1):</u> Since

   $\frac{d}{dx}\left[\frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C\right]=(-\sin x)(\cos x)^{\frac{5}{2}}-(-\sin x)(\cos x)^{\frac{1}{2}}=$

   $\sin x(\cos x)^{\frac{1}{2}}\left[1-\cos^2 x\right]=\sin^3 x\ \sqrt{\cos x}$.

   Therefore, $\displaystyle\int\sin^3 x\ \sqrt{\cos x}\ dx=\frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C$.

   <u>Solution (2):</u> $\displaystyle I=\int\sin^3 x\sqrt{\cos x}\ dx=\int\sin x\sqrt{\cos x}\left(\sin^2 x\right)\ dx=$

   $\displaystyle\int\sin x\sqrt{\cos x}\left(1-\cos^2 x\right)\ dx=\int\sin x\left[(\cos x)^{\frac{1}{2}}-(\cos x)^{\frac{5}{2}}\right]\ dx.$

   Put $u=\cos x$, thus, $du=-\sin x dx\ \Longrightarrow\ \displaystyle I=\int\left(u^{\frac{5}{2}}-u^{\frac{1}{2}}\right)du=\frac{2}{7}u^{\frac{7}{2}}-\frac{2}{3}u^{\frac{3}{2}}+C=$ $\displaystyle\frac{2}{7}(\cos x)^{7/2}-\frac{2}{3}(\cos x)^{3/2}+C.$

6. Let $\displaystyle I_1=\int_0^3\sqrt{3-x}dx$. Put $u=3-x$, thus, $du=-dx, u(0)=3, u(3)=0\ \Longrightarrow$

   $\displaystyle I_1=\int_3^0\sqrt{u}du=\frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^3=\frac{2}{3}(3)^{\frac{3}{2}}=\boxed{2\sqrt{3}}$

   Let $f(x)=x^3\cos x$. Since $f(-x)=-f(x)$, thus, f is odd function and

   $\displaystyle I_2=\int_{-2}^{2}x^3\cos x dx=\boxed{0}$. Then, $\displaystyle\int_0^3\sqrt{3-x}dx+\int_{-2}^{2}x^3\cos x dx=2\sqrt{3}+0=\boxed{2\sqrt{3}}$.

7. $\boxed{f'(x) = \frac{2x}{3+x^4}}$, $\boxed{f''(x) = \frac{6\left(1-x^4\right)}{\left(x^4+3\right)^2} = \frac{6\left(1+x^2\right)(1-x)(1+x)}{\left(x^4+3\right)^2}}$.

f'' changes sign at $x = \pm 1$. Thus, the points of inflection of f are at $x = \pm 1$.

8. Let $f(x) = \sqrt{x^2+16}$, f is increasing ($f'(x) = \frac{x}{\sqrt{x^2+16}} > 0$ for $x \in [0,3]$). Therefore, $f(0) \le f(x) \le f(3)$ for $x \in [0,3]$. [or, $0 \le x \le 3 \implies 0 \le x^2 \le 9 \implies 16 \le x^2 + 16 \le 25 \implies 4 \le \sqrt{x^2+16} \le 5$] Thus, $\int_0^3 f(0)\,dx \le \int_0^3 f(x)\,dx \le \int_0^3 f(3)\,dx \implies$

$$\int_0^3 4dx \le \int_0^3 f(x)\,dx \le \int_0^3 5dx \implies 12 \le \int_0^3 \sqrt{x^2+16}\,dx \le 15.$$

9. The Area $= \int_0^1 \left(y - y^2\right) dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \boxed{\frac{1}{6}}$.

10. (a) Revolution about the line $y = 2$:

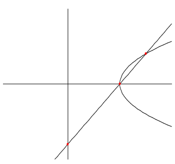
$$Volume = 2\pi \int_0^1 (2-y)\left(y^{\frac{1}{3}} - y^2\right) dy$$

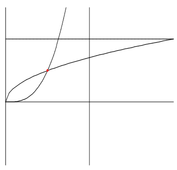
OR $$Volume = \pi \int_0^1 \left[\left(2-x^3\right)^2 - \left(2-\sqrt{x}\right)^2\right] dx.$$

(b) Revolution about the line $x = 2$:

$$Volume = \pi \int_0^1 \left[\left(2-y^2\right)^2 - \left(2-y^{\frac{1}{3}}\right)^2\right] dy$$

OR $$Volume = 2\pi \int_0^1 (2-x)\left(\sqrt{x} - x^3\right) dx.$$





Q9: , Q10: .